Electrical and Electronics Engineering
2024-2025
Master Semester 2

Course
Smart grids technologies

The Optimal Power Flow problem
General aspects

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Outline

Introduction

Recall on convex optimization

Linear programs

Max-Removal Transformation

Non-convexity of the OPF

Introduction

A simple example: a power system **hosts a generic number g of controllable power plants** and we need to **supply u loads (forecasted).**

Let us suppose that the power system is handled by a **vertically integrated utility** (i.e. a monopolistic entity) interested to solve the following problem at a generic time t:

$$\min_{P_{g_1}(t),\dots P_{g_g}(t)} \sum_{i=1}^{g} C_i\left(P_{g_i}(t)\right)$$

$$s.t.$$

$$\sum_{i=1}^{g} P_{g_i}(t) + \sum_{j=1}^{u} P_{l_j}(t) = 0$$

$$P_{g_i}^{min} \leq P_{g_i}(t) \leq P_{g_i}^{max}$$

where

- $P_{g_i}(t)$: is the power output of the generator g_i at time t;
- $P_{g_i}^{min}, P_{g_i}^{max}$: are the min/max power of generator g_i (time indep.);
- $P_{l_j}(t)$: is the (forecast) load of the load l_j at time t;
- $C_i(P_{g_i})$ is the cost of power of generator g_i as function of P_{g_i}.

Introduction

Let us suppose that the grid is modelled as a "**copper plate**", namely **its constraints are disregarded as well as its losses**.

The solution of the previous problem can be determine using the standard **Lagrange multipliers** (for simplicity **we omit the time t and generators constraints**).

$$\mathcal{L} = \sum_{i=1}^{g} C_i\left(P_{g_i}\right) + \lambda\left(\sum_{i=1}^{g} P_{g_i} + \sum_{j=1}^{u} P_{l_j}\right)$$

$$\frac{\partial \mathcal{L}}{\partial P_{g_i}} = 0 \rightarrow \frac{\partial C_1\left(P_{g_1}\right)}{\partial P_{g_1}} + \lambda = 0; \dots ; \frac{\partial C_g\left(P_{g_g}\right)}{\partial P_{g_g}} + \lambda = 0;$$

$$\frac{\partial C_1}{\partial P_{g_1}} = \cdots = \frac{\partial C_g}{\partial P_{g_g}} = -\lambda$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \rightarrow \sum_{i=1}^{g} P_{g_i} + \sum_{j=1}^{u} P_{l_j}$$

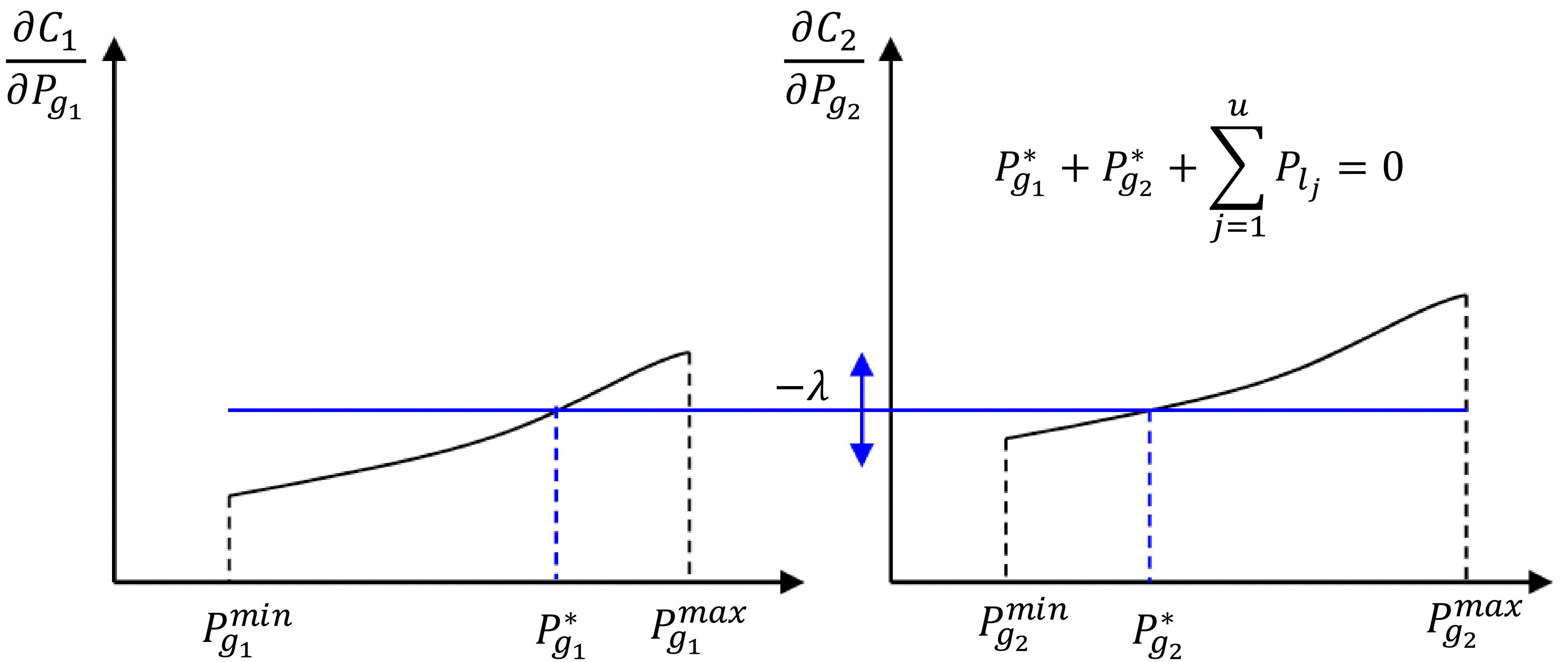
The optimal schedule of the g generators is when $\partial C_i/\partial P_{g_i}$ **are all equal** to λ. They are called **generation units marginal prices**, namely the **variation of the cost of a generation unit as a function of the variation of the power generation of the same unit**.

Introduction

We should also satisfy the g inequalities

$$P_{g_i}^{min} \leq P_{g_i}(t) \leq P_{g_i}^{max}$$

So, graphically we have the following ($g = 2$):



However, **in a real power system there are the grids constraints (i.e. nodal voltage limits and branches power/current flow limits) and controllable resources constraints to be considered.**

Introduction

The **optimal power flow problem (OPF)** has the generic formulation:

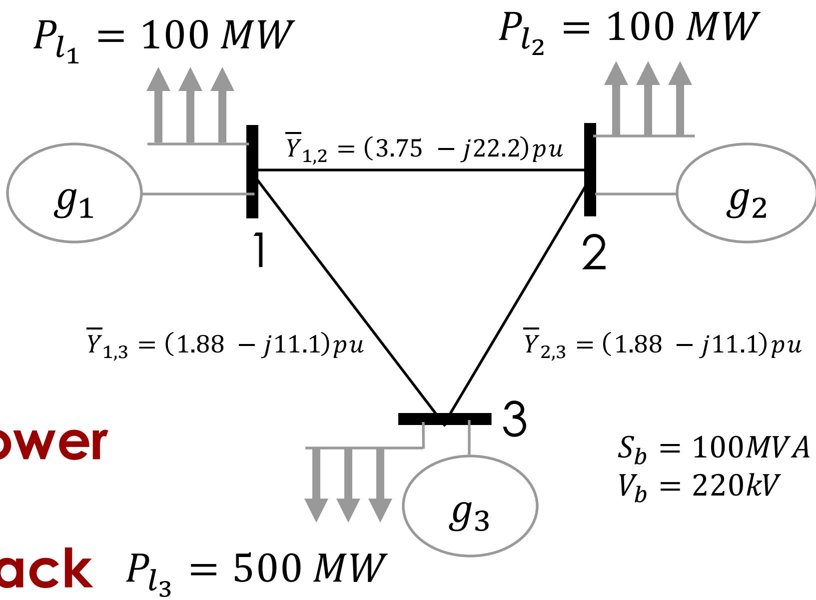
$$\min_{\theta} C$$
$$s.t.$$
$$grid\ constraints$$
$$decision\ variables\ (\theta)\ constraints$$

- C is the cost of the grid operation, not only the one of the generators but also losses and/or other penalty functions.
- Constraints:
  - capability curves of generators, batteries, etc.;
  - nodal voltages within range (usually $0.95 \div 1.05$ of nominal voltage);
  - branches maximum powers or currents (e.g. line ampacity limits);
- Control variables:
  - generators setpoints (active/reactive powers or voltage);
  - reactive power compensators setpoints;
  - batteries setpoints (active/reactive powers);
  - load control setpoints (in case of demand side management);
  - Transformers tap changers or phase shifters.

Introduction

Example of OPF: **determine the optimal active and reactive power setpoints of 3 generators for $1h$ of operation given the knowledge of the grid parameters, constraints and supplied load per node**.

Cost function: $\sum_{i=1,3} C_i\left(P_{g_i}\right)$



Control variables:

- $P_{g_2}, P_{g_3}, Q_{g_2}, Q_{g_3}$

Note that the decision variables are $P_{g_2}, P_{g_3}, Q_{g_2}, Q_{g_3}$ since bus 1 is the slack and its powers are determined by the power balance of the load flow with the other nodal powers fixed. Note also that the slack cost must be in the objective.

Constraints:

- Grid's load flow equations
- Nodal voltage magnitudes within limit
- Branches powers below max
- Generators P^{min}, P^{max} and Q^{min}, Q^{max}.

Quantity	Value
$P_{g_i}^{min}, P_{g_i}^{max}$	$0 \div 400\ MW$
$Q_{g_i}^{min}, Q_{g_i}^{max}$	$-80 \div +80\ MVar$
C_1, C_2, C_3	$15, 1, 225\ CHF/MWh$
$S_{12}^{max}, S_{23}^{max}, S_{31}^{max}$	$200, 200, 300\ MVA$

Introduction

$$\min_{P_{g_2},P_{g_3},Q_{g_2},Q_{g_3}} \sum_{i=1}^{3} C_i\left(P_{g_i}\right)$$

$s.t.$

$$\overline{S}_i = \overline{V}_i \sum_{j=1}^{3} \underline{V_j}\,\underline{Y_{ij}}\,, i = 1,2,3$$

$$\overline{S}_i = \left(P_{g_i} + jQ_{g_i}\right) + \left(P_{l_i} + jQ_{l_i}\right)$$

$$P_{g_i}^{min} \le P_{g_i} \le P_{g_i}^{max},\ i = 1,2,3$$

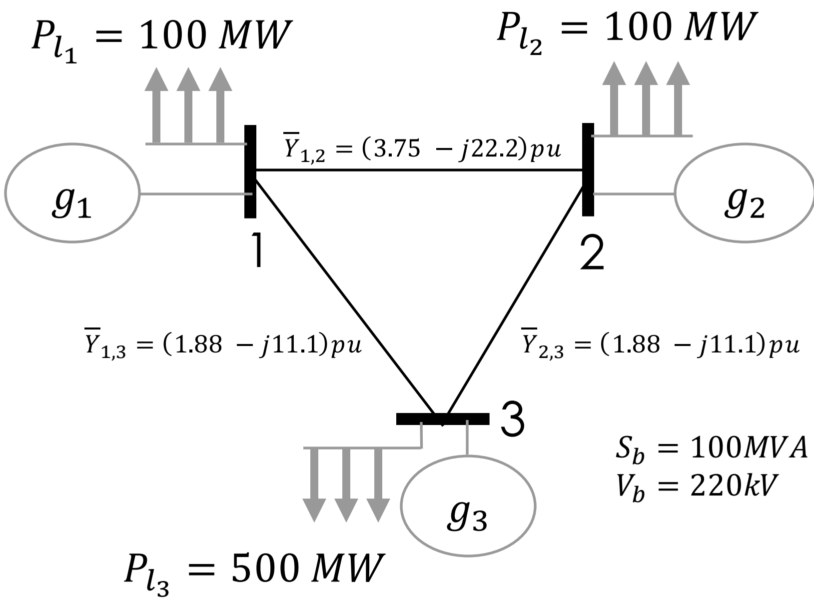
$$Q_{g_i}^{min} \le Q_{g_i} \le Q_{g_i}^{max},\ i = 1,2,3$$

$$\left|\overline{V}_1\right| = 220\text{kV},\ \arg\left(\overline{V}_1\right) = 0;$$

$$V_{min} \le \left|\overline{V}_i\right| \le V_{max}$$

$$\left|\overline{V}_i\right|\left|\overline{Y}_{ij}\left(\overline{V}_i - \overline{V}_j\right)\right| \le S_{i,j}^{max}, i \ne j \quad \text{or} \quad \left|\overline{Y}_{ij}\left(\overline{V}_i - \overline{V}_j\right)\right| \le I_{i,j}^{max}, i \ne j$$

Non-convex problem !



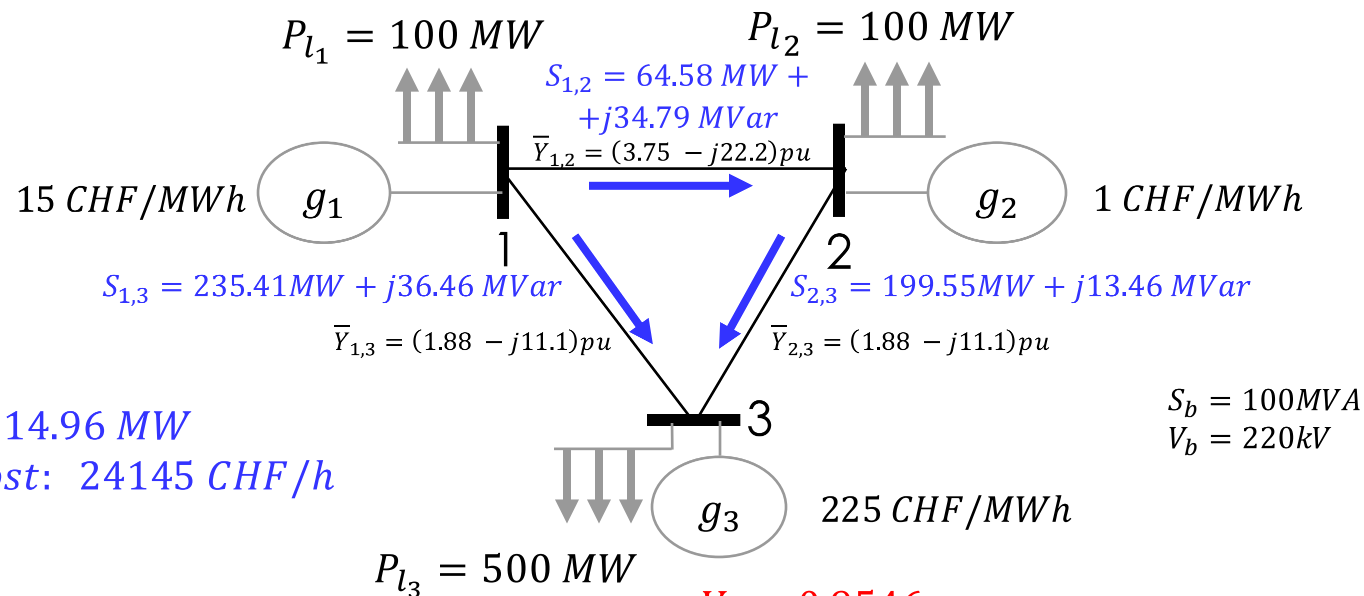
Note that the load powers P_{l_i} can be **either positive or negative** since they are associated to **aggregates of loads and local distributed generation**. So, $P_{l_i} < 0$ for the case of a load.

Introduction

Solution

$V_1 = 1\ pu$
$\theta_1 = 0\ mrad$
$\lambda_{P_1} = 60.04\ CHF/MWh$
$P_{g_1} + j\ Q_{g_1} = 400MW + j80MVar$

$V_2 = 0.9803\ pu$
$\theta_2 = -26.23\ mrad$
$\lambda_{P_2} = 1\ CHF/MWh$
$P_{g_2} + j\ Q_{g_2} = 235.36MW - j10.27MVar$



$Losses$: $14.96\ MW$
$Total\ cost$: $24145\ CHF/h$

$V_3 = 0.9546\ pu$
$\theta_3 = -211.97\ mrad$
$\lambda_{P_3} = 226.94\ CHF/MWh$
$P_{g_3} + j\ Q_{g_3} = 79.6MW + j80MVar$

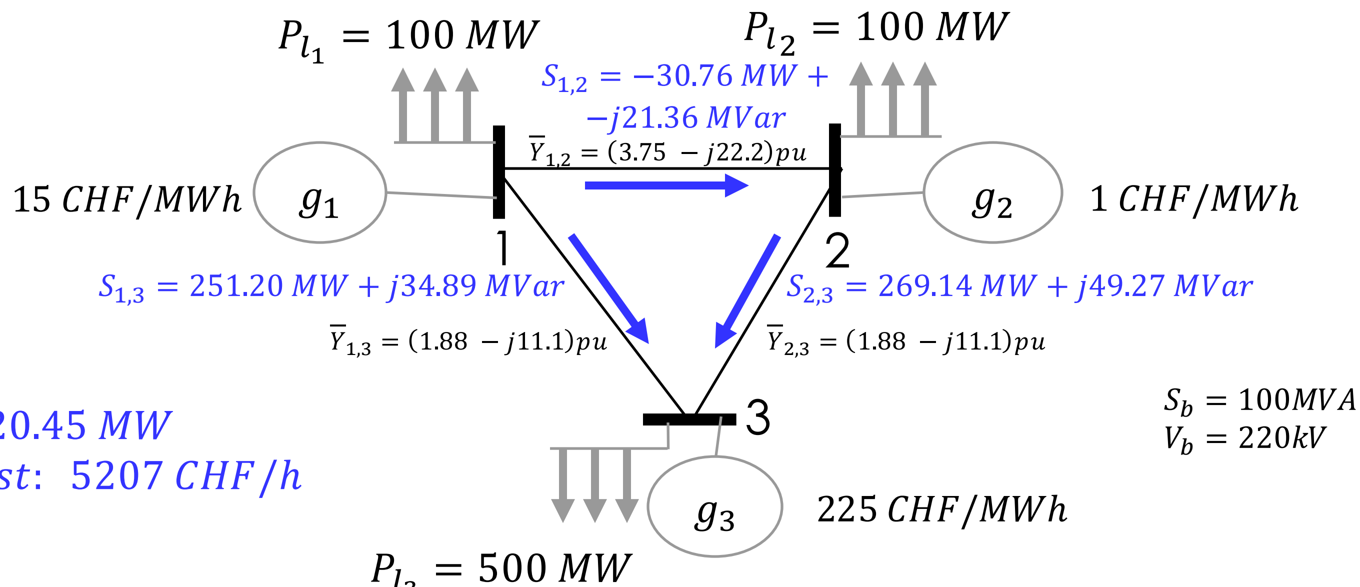
Quantity	Value
$P_{g_i}^{min}, P_{g_i}^{max}$	$0 \div 400\ MW$
$Q_{g_i}^{min}, Q_{g_i}^{max}$	$-80 \div +80\ MVar$
C_1, C_2, C_3	$15, 1, 225\ CHF/MWh$
$S_{12}^{max}, S_{23}^{max}, S_{13}^{max}$	$200, 200, 300\ MVA$

Introduction

Solution

$V_1 = 1\ pu$
$\theta_1 = 0\ mrad$
$\lambda_{P,1} = 15\ CHF/MWh$
$P_{g_1} + j\ Q_{g_1} = 320.45\ MW + j22.27\ MVar$

$V_2 = 1.0117\ pu$
$\theta_2 = +11.75\ mrad$
$\lambda_{P,2} = 14.96\ CHF/MWh$
$P_{g_2} + j\ Q_{g_2} = 400MW + j80MVar$



Losses: 20.45 *MW*
Total cost: 5207 *CHF/h*

$V_3 = 0.9567\ pu$
$\theta_3 = -226.55\ mrad$
$\lambda_{P,3} = 16.36\ CHF/MWh$
$P_{g_3} + j\ Q_{g_3} = 0MW + j80MVar$

Quantity	Value
$P_{g_i}^{min}, P_{g_i}^{max}$	$0 \div 400\ MW$
$Q_{g_i}^{min}, Q_{g_i}^{max}$	$-80 \div +80\ MVar$
C_1, C_2, C_3	$15, 1, 225\ CHF/MWh$
$S_{12}^{max}, S_{23}^{max}, S_{13}^{max}$	$2000, 2000, 3000\ MVA$

Lines power transmission limits x10

Introduction

By **concatenating the previous problem for each time slot of the day**, we obtain the **schedule of each generator**. This type of problems are called **dispatching problems**. In this case, we need to also add a **time-dependency constraint** associated to generator **ramping** along with the fact that also **other constraints are time-dependent**.

$$\min_{P_{g_2}(t),\dots,P_{g_g}(t),Q_{g_1}(t),\dots,Q_{g_g}(t)} \sum_{t=1}^{24}\sum_{i=1}^{g} C_i\left(P_{g_i}(t)\right)$$

$s.t.$

$$\bar{S}_i(t) = \bar{V}_i(t)\sum_{j=1}^{s} \underline{V}_j(t)\underline{Y}_{ij}\,, i = 1, \dots, s$$

$$\bar{S}_i(t) = \left(P_{g_i}(t) + jQ_{g_i}(t)\right) + \left(P_{l_i}(t) + jQ_{l_i}(t)\right), i = 1, \dots, s$$

$$P_{g_i}^{min} \le P_{g_i}(t) \le P_{g_i}^{max}, i = 1, \dots, g$$

$$Q_{g_i}^{min} \le Q_{g_i}(t) \le Q_{g_i}^{max}, i = 1, \dots, g$$

$$|\bar{V}_1| = 1pu,\ \arg(\bar{V}_1) = 0;$$

$$V_{min} \le |\bar{V}_i(t)| \le V_{max}, i = 2, \dots, s$$

$$|\bar{V}_i(t)|\left|\bar{Y}_{ij}\left(\bar{V}_i(t) - \bar{V}_j(t)\right)\right| \le S_{i,j}^{max}, \text{or } \left|\bar{Y}_{ij}\left(\bar{V}_i(t) - \bar{V}_j(t)\right)\right| \le I_{i,j}^{max}, i \ne j = 1, \dots, s$$

$$\xi_{g_i}^{min} \le P_{g_i}(t+1) - P_{g_i}(t) \le \xi_{g_i}^{max}$$

Where s is the nr. of grid's nodes and g the nr. of generation units.

Introduction

In case some generators are **energy storage systems**, we need to enforce that their **energy content** (e.g. the **state-of-charge of a battery**) **are within bounds**.

$$\min_{\substack{P_{g_2}(t),\dots,P_{g_g}(t),Q_{g_2}(t),\dots,Q_{g_g}(t)\\ P_{s_1}(t),\dots,P_{s_m}(t),Q_{s_1}(t),\dots,Q_{s_m}(t)}} \sum_{t=1}^{T_{max}} \left(\sum_{i=1}^{g} C_i\left(P_{g_i}(t)\right) + \sum_{i=1}^{m} C_i\left(P_{s_i}(t)\right) \right)$$

$s.t.$

$$\overline{S}_i(t) = \overline{V}_i(t) \sum_{j=1}^{n} \underline{V}_j(t) \underline{Y}_{ij}, i = 1, \dots, s$$

$$\overline{S}_i(t) = \left(P_{g_i}(t) + jQ_{g_i}(t)\right) + \left(P_{s_i}(t) + jQ_{s_i}(t)\right) + \left(P_{l_i}(t) + jQ_{l_i}(t)\right), i = 1, \dots, s$$

$$P_{gi}^{min} \le P_{g_i}(t) \le P_{gi}^{max}, i = 1, \dots, g$$

$$Q_{gi}^{min} \le Q_{g_i}(t) \le Q_{gi}^{max}, i = 1, \dots, g$$

$$P_{si}^{min} \le P_{s_i}(t) \le P_{si}^{max}, i = 1, \dots, m$$

$$Q_{si}^{min} \le Q_{s_i}(t) \le Q_{si}^{max}, i = 1, \dots, m$$

$$\left|\overline{V}_1\right| = 1pu, \ \arg\left(\overline{V}_1\right) = 0;$$

$$V_{min} \le \left|\overline{V}_i(t)\right| \le V_{max}, i = 2, \dots, s$$

$$\left|\overline{V}_i(t)\right| \left|\overline{Y}_{ij}\left(\overline{V}_i(t) - \overline{V}_j(t)\right)\right| \le S_{i,j}^{max}, \text{or } \left|\overline{Y}_{ij}\left(\overline{V}_i(t) - \overline{V}_j(t)\right)\right| \le I_{i,j}^{max}, i \ne j = 1, \dots, s$$

$$\xi_{gi}^{min} \le P_{g_i}(t+1) - P_{g_i}(t) \le \xi_{gi}^{max}$$

$SoC_i(t+1) = SoC_i(t) + P_{s_i}(t+1)\Delta t, \ i = 1, \dots, m$ (lossless model of the storage device i)

$$SoC_i^{min} \le SoC_i(t+1) \le SoC_i^{max}, i = 1, \dots, m$$

Where m is the number of energy storage devices.

Introduction

In case we add start-up and shutdown costs of the generators, we obtain the so-called **unit commitment problem.**

$$\min_{\substack{P_{g_2}(t),\ldots,P_{g_g}(t),Q_{g_2}(t),\ldots,Q_{g_g}(t)\\ w_{g_1}(t),\ldots,w_{g_g}(t)}} \sum_{t=1}^{24}\left(\sum_{i=1}^{g} C_i\left(P_{g_i}(t)\right) + SU_{g_i}(t)w_{g_i}(t)\left(1-w_{g_i}(t-1)\right) + SD_{g_i}(t)\left(1-w_{g_i}(t)\right)w_{g_i}(t-1)\right)$$

$s.t.$

$w_{g_1}(t) \in \{0,1\}$

$\overline{S}_i(t) = \overline{V}_i(t)\sum_{j=1}^{n} \underline{V}_j(t)\underline{Y}_{ij}\,, i = 1,\ldots,s$

$\overline{S}_i(t) = \left(P_{g_i}(t) + jQ_{g_i}(t)\right) + \left(P_{l_i}(t) + jQ_{l_i}(t)\right), i = 1,\ldots,s$

$P_{g_i}^{min} \le P_{g_i}(t) \le P_{g_i}^{max}, i = 1,\ldots,g$

$Q_{g_i}^{min} \le Q_{g_i}(t) \le Q_{g_i}^{max}, i = 1,\ldots,g$

$|\overline{V}_1|$=1pu, $\arg(\overline{V}_1) = 0;$

$V_{min} \le |\overline{V}_i(t)| \le V_{max}, i = 2,\ldots,s$

$|\overline{V}_i(t)|\left|\overline{Y}_{ij}\left(\overline{V}_i(t) - \overline{V}_j(t)\right)\right| \le S_{i,j}^{max}$, or $\left|\overline{Y}_{ij}\left(\overline{V}_i(t) - \overline{V}_j(t)\right)\right| \le I_{i,j}^{max}, i \ne j = 1,\ldots,s$

$\xi_{g_i}^{min} \le P_{g_i}(t+1) - P_{g_i}(t) \le \xi_{g_i}^{max}$

where $SU_i(t)$ is the cost of **starting up** unit i at time t, $SD_i(t)$ is the cost of **shutting it down** and $w_{g_1}(t)$ the i**nteger variable associated to the state of the generation unit**: $w_i(t) = 1 \Leftrightarrow P_{g_i}(t) > 0$.

Outline

Introduction

Recall on convex optimization

Linear programs

Max-Removal Transformation

Non-convexity of the OPF

Recall on convex optimization

The OPF problem is about optimisation.

Convex optimization problems are **continuous** optimization problems and are usually **tractable** (i.e., can be solved exactly for large dimensions, up to hundreds of thousands of dimensions).

Non-convex complex continuous optimization problems can be very hard to solve exactly, even for modest dimensions; they are solved approximately using heuristics that often need an initial guess.

Observation: **continuous** means that the optimization variables are real or complex numbers – as opposed to "discrete" optimization problems where the optimization variables can be represented as integers.

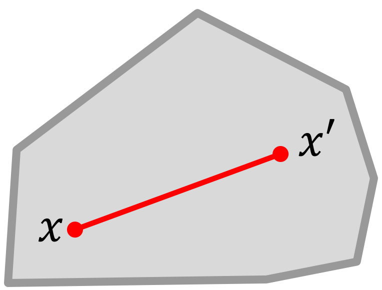
In the context of OPF problems, we can have **discrete OPF** in case, for instance, we would like to **determine which generators are on or off at a given time** t accounting for their **start-up and shutdown costs.** This type of problem is called **unit commitment (see slide 13)**.

Recall on convex optimization

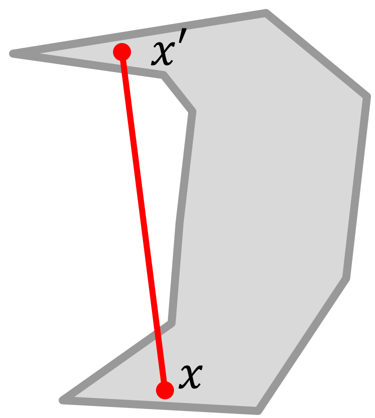
Convex optimization problems

- $\min f_0(x)$
  over all $x \in X$
- where X is a convex subset of $\mathbb{R}^n$ and f_0 is a convex function

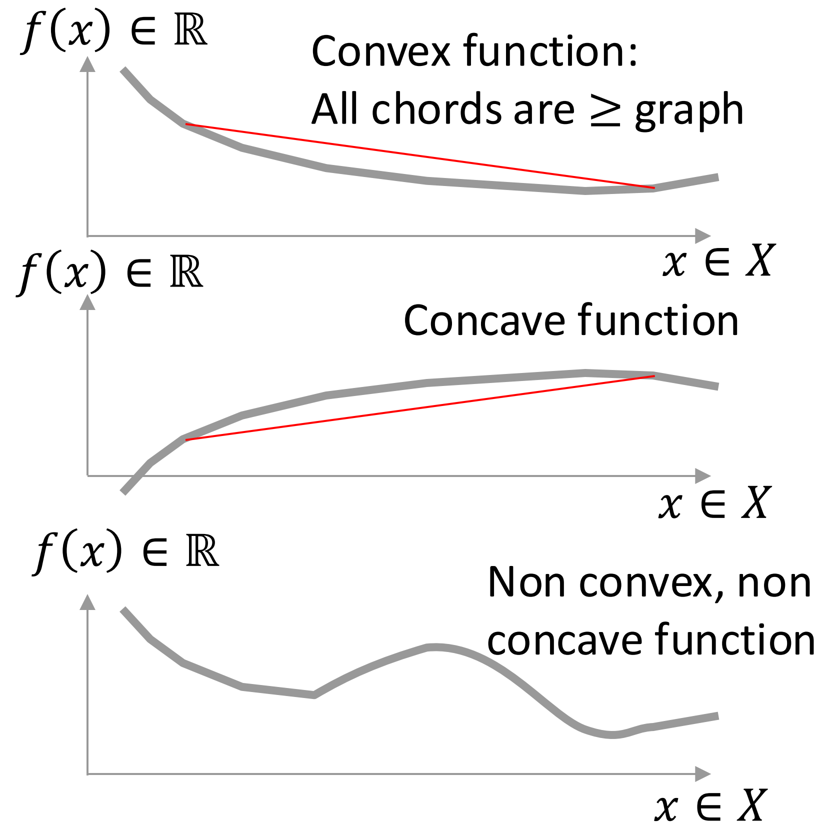
- $\max f_0(x)$
  over all $x \in X$
- where X is a convex subset of $\mathbb{R}^n$ and f_0 is a concave function



Convex set
$\forall x, x' \in X, [x, x'] \subset X$



Non-convex set
$\exists x, x' \in X, [x, x'] \not\subset X$



Recall on convex optimization

How to test convexity ?

1. Function f is convex if the domain $\operatorname{dom} f$ is convex and if $\forall x, y \in \operatorname{dom} f$ and $\lambda \in [0,1]$ we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

2. If f is differentiable then f is convex iff $f(y) \geq f(x) + \nabla f^T(x)(y-x)$ $\forall x, y \in \operatorname{dom} f$.

3. If f is twice differentiable then f is convex iff $\nabla^2 f(x) \succcurlyeq 0$ (positive semidefinite Hessian).

4. If $f_i, i \in I$ are convex and $c_i \geq 0$, then $f = \sum_{i \in I} c_i f_i$ is convex.

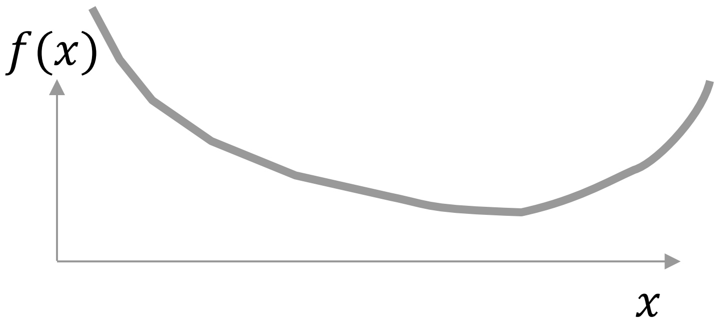
5. Under the same assumptions $f(x) = \max_i f_i(x)$ is convex.

6. If f, h are convex functions and h is increasing, then $g = h(f(\cdot))$ is convex.

7. The set $\{x: f(x) \leq c\}$ is convex if f is convex (level set).

Recall on convex optimization

Why are convex problems special ?



$\min f(x)$ is a convex problem
any local minimum
is a global minimum.

$\min f(x)$ is not a convex problem
there can be **many local minima.**

Recall on convex optimization

Which of these problems is convex ?

1. A
2. B
3. C
4. A and B
5. A and C
6. B and C
7. All
8. None
9. I don't know

A: max x^2 over $x \in [a, b]$

B: min x^2 over $x \in [a, b]$

C: max $x + y$ over $(x, y) \in \mathbb{R}^2$ subject to

$$x + 2y \leq 10$$
$$2x + y \leq 8$$
$$x \geq 0, y \geq 0$$

Recall on convex optimization

Which of these problems is convex ?

1. A
2. B
3. C
4. A and B
5. A and C
6. B and C
7. All
8. None
9. I don't know

A: max x^2 over $x \in [a, b]$

B: min x^2 over $x \in [a, b]$

C: max $x + y$ over $(x, y) \in \mathbb{R}^2$ subject to

$$x + 2y \leq 10$$
$$2x + y \leq 8$$
$$x \geq 0, y \geq 0$$

Answer 6

A is not a convex optimization problem; this is a maximization and the function should be concave, which is not true.

B is a convex optimization problem: the function to be minimized is convex and the set X is an interval, which is convex

C is a a convex optimization problem: the function to be maximized is convex (and concave) and the set X is defined by linear inequalities, therefore is convex.

Recall on convex optimization

Is this problem convex ?

$$\min_{x,y} ax^2 + by^2$$
$$s.t. \quad x^2 + y^2 \leq c^2$$

1. Yes
2. No
3. Depends on a and b
4. Depends on a, b, and c
5. I don't know

Recall on convex optimization

Is this problem convex ?

$$\min_{x,y} ax^2 + by^2$$
$$s.t. \quad x^2 + y^2 \leq c^2$$

1. Yes
2. No
3. Depends on a and b
4. Depends on a, b, and c
5. I don't know

Answer 3

The constraint $x^2 + y^2 \leq c^2$ always defines a convex set irrespective of the value of c.

Since the objective function $f(x, y) = ax^2 + by^2$ is twice differentiable, it is convex if and only if the Hessian $\nabla^2 f$ is positive semidefinite (PSD). In this case, the Hessian matrix is $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, which is PSD only when $a, b \geq 0$.

For example, $x^2 + y^2$ is a convex function of (x, y) but $x^2 - y^2$ is not.

Recall on convex optimization

Is this problem convex ?

$$\min_{x,y} \; x^2 - y^2$$
$$s.t. \quad x^2 + y^2 = c^2 \; (c > 0)$$

1. Yes
2. No
3. It depends on c
4. I don't know

Recall on convex optimization

Is this problem convex ?

$$\min_{x,y} \; x^2 - y^2$$
$$s.t. \quad x^2 + y^2 = c^2 \; (c > 0)$$

1. Yes
2. No
3. It depends on c
4. I don't know

Answer 2

The set $x^2 + y^2 = c^2$ represents a circle in two dimensions. It is not a convex set.

Recall on convex optimization

Is it possible to **reformulate** the problem as a convex problem?

$$\min_{x,y} \; x^2 - y^2$$
$$s.t. \quad x^2 + y^2 = c^2$$

1. Yes
2. No
3. It depends on c
4. I don't know

Recall on convex optimization

Is it possible to **reformulate** the problem as a convex problem?

$$\min_{x,y} \; x^2 - y^2$$
$$s.t. \quad x^2 + y^2 = c^2$$

1. Yes
2. No
3. It depends on c
4. I don't know

Answer 1

We can eliminate y in the objective function by using the constraint:

$$(x^2 + y^2 = c^2) \Leftrightarrow y = \pm\sqrt{c^2 - x^2} \text{ and } |x| \leq |c|$$

Thus, the problem is equivalent to: $\min_x \left(x^2 - (c^2 - x^2)\right) \quad \text{s.t.} \, |x| \leq |c|$

That is equivalent to $\min \; 2x^2 - c^2 \quad \text{s.t.} \, |x| \leq |c|$ which is a convex problem.

Take home message: transformation of problem formulation is an important topic. Some non-convex formulations can be made convex, but others can't.

Outline

Introduction

Recall on convex optimization

Linear programs

Max-Removal Transformation

Non-convexity of the OPF

Linear programs

A linear program can be described in the form:

$$\min_{x \in \mathbb{R}^n} c^T x$$
$$s.t.$$
$$Ax \leq b, Cx = d$$

Indeed, the objective function is linear (so, it is convex) and the set $X \subset \mathbb{R}^n$ is defined by linear equalities and inequalities (that are convex too). Here $c \in \mathbb{R}^n$ and A, C are matrices with n columns.

For this type of optimizations, there exists very efficient packages to solve large scale problems.

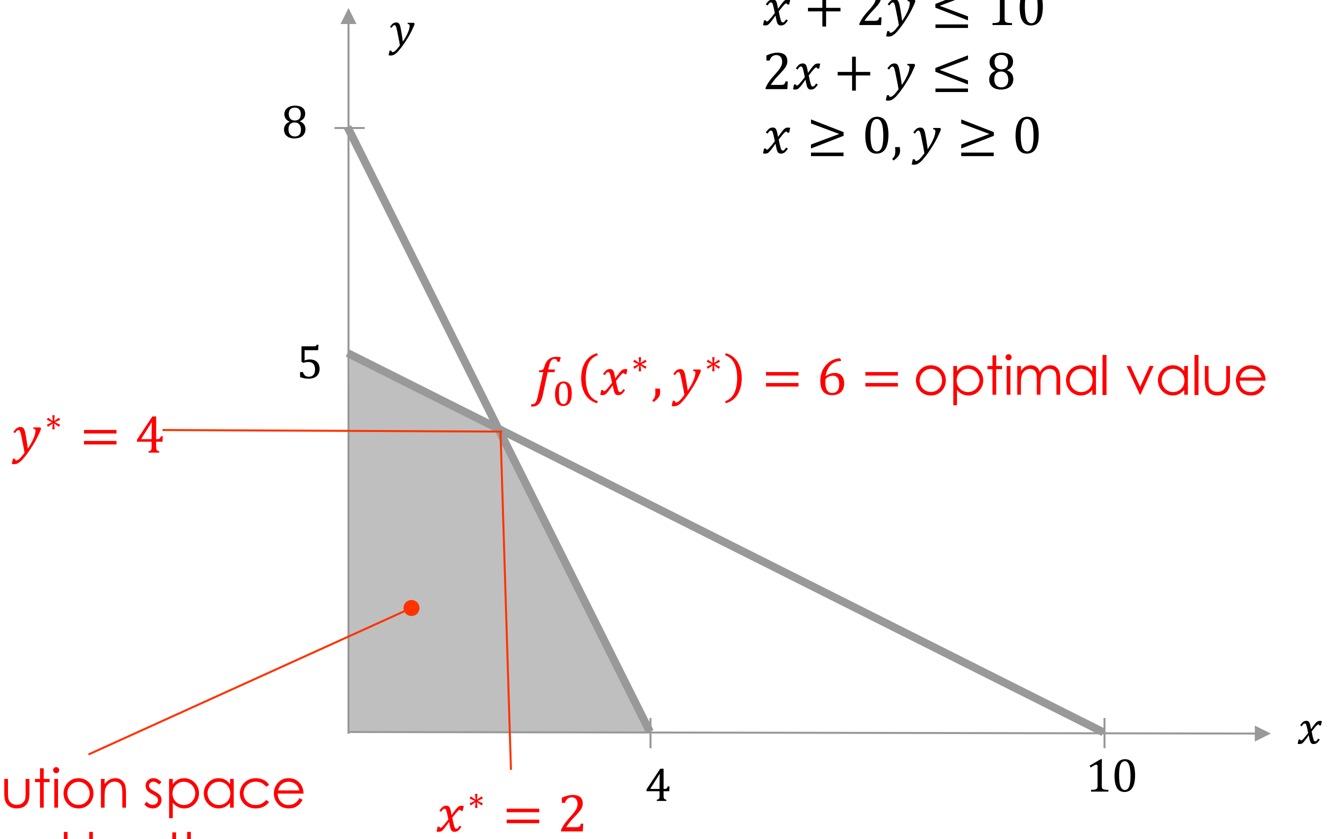
Note that the following problem (obtained by changing c into $-c$) is also a linear program

$$\max_{x \in \mathbb{R}^n} c^T x$$
$$s.t.$$
$$Ax \leq b, Cx = d$$

Linear programs

Example

$$\max f_0(x) = x + y$$
$$s.t.$$
$$x + 2y \leq 10$$
$$2x + y \leq 8$$
$$x \geq 0, y \geq 0$$



The solution space identified by the constraints is convex (grey area)

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Max-Removal Transformation

Which problem can be formulated as LP?

1. A
2. B
3. Both
4. None
5. I don't know

A: $\min_{x,y} (x^2 + y^2)$

$s.t.$

$x + 2y \leq 10$

$2x + y \leq 8$

$x \geq 0, y \geq 0$

B: $\min_{x,y} f_0(x, y)$

$s.t.$

$x + 2y \leq 10$

$2x + y \leq 8$

$x \geq 0, y \geq 0$

where $f_0(x, y) := \max(x + y, 2y)$

Max-Removal Transformation

Which problem can be formulated as LP?

1. A
2. B
3. Both
4. None
5. I don't know

A: $\min_{x,y} (x^2 + y^2)$

$s.t.$

$x + 2y \leq 10$

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$x \geq 0, y \geq 0$

B: $\min_{x,y} f_0(x, y)$

$s.t.$

$x + 2y \leq 10$

$2x + y \leq 8$

$x \geq 0, y \geq 0$

where $f_0(x, y) := \max(x + y, 2y)$

Answer 2

A is a convex optimization problem but is not a linear program. It does not seem possible to transform it exactly into an equivalent linear program.

B is not formulated as a linear program, but, as we show next, it is equivalent to a linear program.

Max-Removal Transformation

The problem B is equivalent to problem B'

Problem B: $\min_{x,y} f_0(x,y)$

$s.t.$

$x + 2y \leq 10$

$2x + y \leq 8$

$x \geq 0, y \geq 0$

where $f_0(x,y) := \max(x + y, 2y)$

Problem B': $\min_{t,\, x,y} t$

$s.t.$

$t \geq x + y$

$t \geq 2y$

$x + 2y \leq 10$

$2x + y \leq 8$

$x \geq 0, y \geq 0$

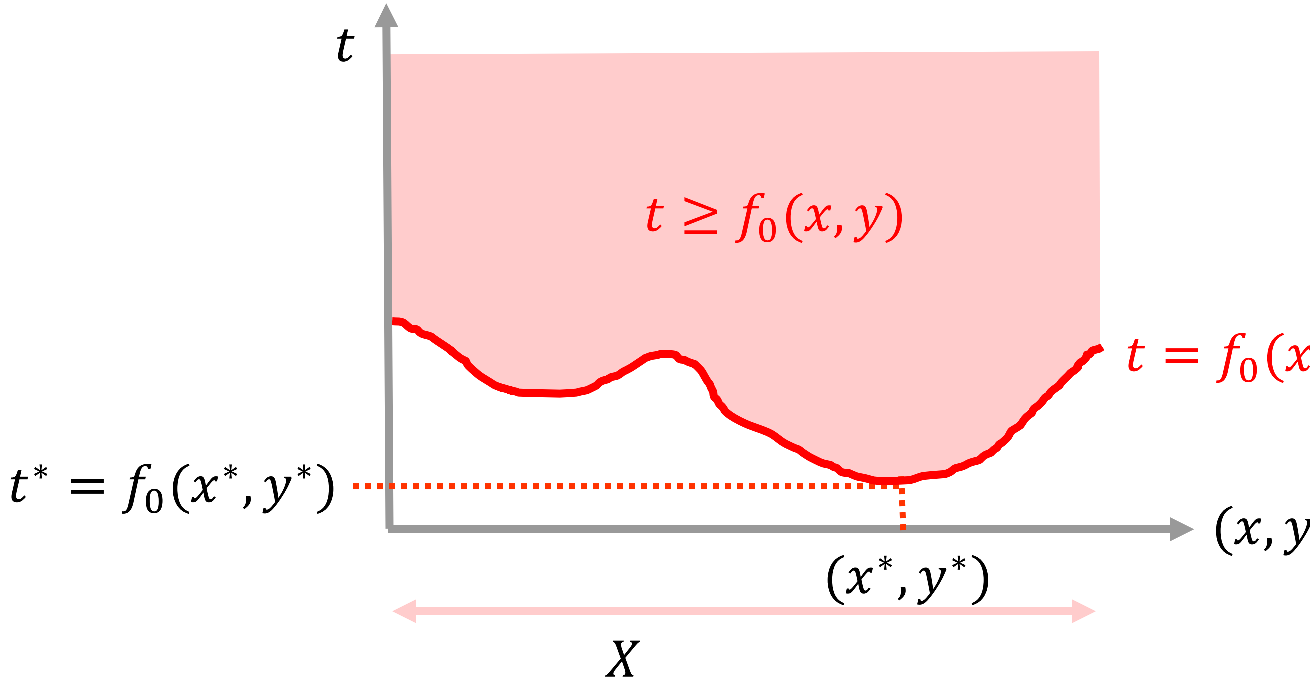
Let us see why $B \Leftrightarrow B'$: the key observation is that the constraints

$t \geq x + y, t \geq 2y$ are equivalent to $t \geq \max(x + y, 2y)$.

Max-Removal Transformation

Let X be the set of (x, y) that are feasible for problem B, i.e. $X = \{(x \geq 0\,, y \geq 0\,): x + 2y \leq 10, 2x + y \leq 8\}$. Thus, problem B' can be re-written as

$$\min_{t,x,y}\ t$$
$$s.t.$$
$$t \geq f_0(x, y)$$
$$(x, y) \in X$$



The optimal value of B' is the $\min$ of t in the shaded area. We see on the figure that it is the minimum of $f_0(x, y)$ over $(x, y) \in X$,which is the optimum of B.

Max-Removal Transformation

Any problem of the form

$$\min_x f_1(x) + \ \dots + f_n(x) + h(x)$$
$$s.t.$$
$$x \in X \subset \mathbb{R}^m$$

where $f_j(x) \coloneqq \max_{i=1\dots n_j} g_{j,i}(x)$

is equivalent to

$$\min_{t,x} t_1 + \cdots + t_n + h(x)$$
$$s.t.$$
$$x \in X, t = (t_1, \dots t_n) \in \mathbb{R}^n$$
$$t_j \geq g_{j,i}(x), \forall i = 1 \dots n_j, \forall j = 1 \dots n$$

We call this process the **max-removal transformation** as it removes the $\max$ terms from the objective function at the expense of adding one optimization variable per max. It is very often used in the context of OPFs.

Max-Removal Transformation

Which problem is a re-formulation of problem P as a Linear Program ?

1. A
2. B
3. Both
4. None
5. I don't know

(P)

$$\min\ |x - y - 5|\ \text{over}$$
$$x + 2y \leq 10$$
$$2x + y \leq 8$$
$$x \geq 0, y \geq 0$$

(A)

$$\min t\ \text{over}$$
$$t \geq x - y - 5$$
$$t \geq -x + y + 5$$
$$x + 2y \leq 10$$
$$2x + y \leq 8$$
$$x \geq 0, y \geq 0$$

(B)

$$\min\ t\ \text{over}$$
$$t \leq x - y - 5$$
$$t \leq -x + y + 5$$
$$x + 2y \leq 10$$
$$2x + y \leq 8$$
$$x \geq 0, y \geq 0$$

Max-Removal Transformation

Which problem is a re-formulation of problem P as a Linear Program ?

1. A
2. B
3. Both
4. None
5. I don't know

(P)
$\min\ |x-y-5|$ over
$x+2y \leq 10$
$2x+y \leq 8$
$x \geq 0, y \geq 0$

(A)
$\min t$ over
$t \geq x-y-5$
$t \geq -x+y+5$
$x+2y \leq 10$
$2x+y \leq 8$
$x \geq 0, y \geq 0$

(B)
$\min\ t$ over
$t \leq x-y-5$
$t \leq -x+y+5$
$x+2y \leq 10$
$2x+y \leq 8$
$x \geq 0, y \geq 0$

Answer 1

Recall that the absolute function can be rewritten as follows:

$$|x| = \max(x, -x)$$

Therefore, the objective function $|x-y-5|$ can be written as a maximum of linear functions:

$$|x-y-5| = \max(x-y-5, -x+y+5)$$

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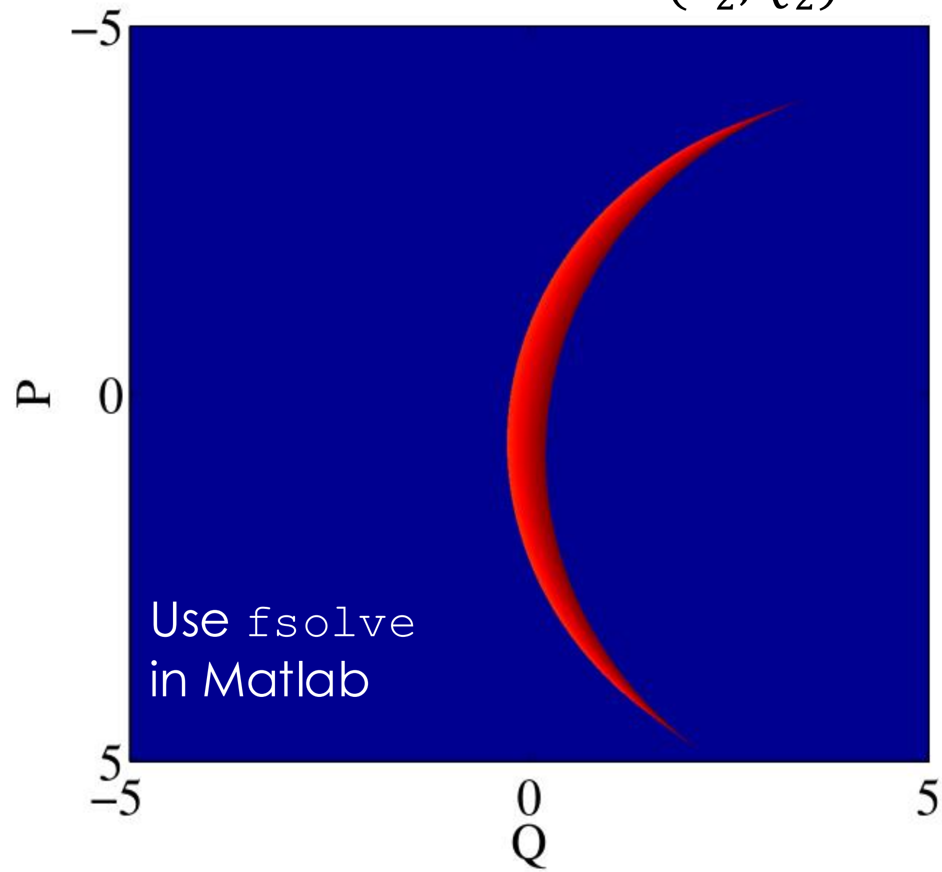
Non-convexity of the OPF

By looking at the original AC-OPF problem, we can say that it is a non-convex optimization problem because the **set X of feasible variables is not-convex.** Let's look at a simple example.

(P_2, Q_2) decision variables

Feasible set (P_2, Q_2)



Line admittance:

$$\overline{Y}_{1,2} = 0.734 - j\,4.890\,p.u.$$

Node 1 is the slack bus:

$$\overline{V}_1 = 1 + j0\,pu$$

Constraints on voltage magnitude at node 2

$$0.95 \le |\overline{V}_2| \le 1.05\,pu$$

So, the feasible set for (P_2, Q_2) is given by:

$$\overline{S}_{2,1} = P_2 + jQ_2 = \overline{V}_2 \underline{Y}_{1,2}(\underline{V}_2 - \underline{V}_1)$$

$$\sqrt{(V_2^{re})^2 + (V_2^{im})^2} = const \in [0.95, 1.05]$$

Non-convexity of the OPF

In the next lectures we will use two approaches to render the OPF problem convex:

1. replace the original constraints by means of some linear approximations;
2. relax the original set of constraints.